REGULATOR OF MODULAR UNITS AND MAHLER MEASURES

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Abstract. We present a proof of the formula, due to Mellit and Brunault, which evaluates an integral of the regulator of two modular units to the value of the L-series of a modular form of weight 2 at $s = 2$. Applications of the formula to computing Mahler measures are discussed.

1. Introduction

The work of C. Deninger [7], D. Boyd [3], F. Rodriguez-Villegas [13] and others provided us with a natural link between the (logarithmic) Mahler measures

$$\mathrm{m}\big(P(x_1,\dots,x_m)\big) := \frac{1}{(2\pi i)^m} \int\!\!\cdots\!\!\int_{|x_1|=\cdots=|x_m|=1} \log|P(x_1,\dots,x_m)| \,\frac{\mathrm{d}x_1}{x_1}\cdots\frac{\mathrm{d}x_m}{x_m}$$

of certain (Laurent) polynomials $P(x_1,\dots,x_m)$, higher regulators and Beĭlinson's conjectures, though it took a while for those original ideas to become proofs of some conjectural evaluations of Mahler measures. In this note we mainly discuss a recent general formula for the regulator of two modular units due to A. Mellit and F. Brunault, its consequences for 2-variable Mahler measures and some related problems.

For a smooth projective curve C given as the zero locus of a polynomial $P(x,y) \in \mathbb{C}[x,y]$ and two rational non-constant functions g and h on C, define the 1-form

$$\eta(g,h) := \log|g|\,\mathrm{d}\arg h - \log|h|\,\mathrm{d}\arg g; \tag{1}$$

here $\mathrm{d}\arg g$ is globally defined as $\mathrm{Im}(\mathrm{d}g/g)$. The form (1) is a real 1-form defined and infinitely many times differentiable on $C\setminus S$, where S is the set of zeros and poles of g and h. Furthermore, it is not hard to verify that the form (1) is antisymmetric, bi-additive and closed; the latter fact follows from

$$\mathrm{d}\eta(g,h) = \mathrm{Im}\bigg(\frac{\mathrm{d}g}{g}\wedge\frac{\mathrm{d}h}{h}\bigg) = 0,$$

Date: 14 April 2013. *Revised*: 23 September 2013.

2010 *Mathematics Subject Classification.* Primary 11F67; Secondary 11F11, 11F20, 11G16, 11G55, 11R06, 14H52, 19F27.

Key words and phrases. Regulator, Mahler measure, L-value of elliptic curve.

Work is supported by the Australian Research Council.

as the curve C has dimension 1. In turn, the closedness of (1) implies that, for a closed path γ in $C \setminus S$, the regulator map

$$r(\{g,h\})\colon \gamma \mapsto \int_\gamma \eta(g,h) \tag{2}$$

only depends on the homology class $[\gamma]$ of γ in $H_1(C\setminus S,\mathbb{Z})$.

Assuming that the polynomial $P(x,y)$ is tempered [2, 13], factorising it as a polynomial in y with coefficients from $\mathbb{C}[x]$,

$$P(x,y) = a_0(x)\prod_{j=1}^{n}(y-y_j(x)),$$

and applying Jensen's formula, we can write [2, 6, 10, 13] the Mahler measure of P in the form

$$\mathrm{m}\big(P(x,y)\big) = \mathrm{m}\big(a_0(x)\big) + \frac{1}{2\pi}\,r(\{x,y\})([\gamma]), \tag{3}$$

where

$$\gamma := \bigcup_{j=1}^{n}\big\{(x,y_j(x)) : |x|=1,\ |y_j(x)|\ge 1\big\} = \{(x,y)\in C : |x|=1,\ |y|\ge 1\} \tag{4}$$

is the union of at most n closed paths in $C\setminus S$.

In case the curve $C : P(x,y)=0$ admits a parameterisation by means of modular *units* $x(\tau)$ and $y(\tau)$, where the modular parameter τ belongs to the upper halfplane $\mathbb{H}=\{\tau\in\mathbb{C} : \operatorname{Im}\tau>0\}$, one can change to the variable τ in the integral (2) for $r(\{x,y\})$; the class $[\gamma]$ in this case [5] becomes a union of paths joining certain cusps of the modular functions $x(\tau)$ and $y(\tau)$. The following general result completes the computation of the Mahler measure in the case when $x(\tau)$ and $y(\tau)$ are given as quotients/products of modular units

$$g_a(\tau) := q^{NB(a/N)/2}\prod_{\substack{n\ge1\\ n\equiv a \bmod N}}(1-q^n)\prod_{\substack{n\ge1\\ n\equiv -a \bmod N}}(1-q^n), \quad q=\exp(2\pi i\tau), \tag{5}$$
$$\text{where}\quad B(x)=B_2(x) := \{x\}^2-\{x\}+\tfrac16.$$

Theorem 1 (Mellit–Brunault [12])**.** *For a, b and c integral, with ac and bc not divisible by N,*

$$\int_{c/N}^{i\infty}\eta(g_a,g_b) = \frac{1}{4\pi}\,L(f(\tau)-f(i\infty),2), \tag{6}$$

where the weight 2 modular form $f(\tau)=f_{a,b;c}(\tau)$ is given by

$$f_{a,b;c} := e_{a,bc}e_{b,-ac} - e_{a,-bc}e_{b,ac}$$

and

$$e_{a,b}(\tau) := \frac12\bigg(\frac{1+\zeta_N^a}{1-\zeta_N^a}+\frac{1+\zeta_N^b}{1-\zeta_N^b}\bigg)+\sum_{m,n\ge1}(\zeta_N^{am+bn}-\zeta_N^{-(am+bn)})q^{mn}, \quad \zeta_N := \exp(2\pi i/N), \tag{7}$$

are weight 1 level N^2 Eisenstein series.

The L-value on the right-hand side of (6) is well defined because of subtracting the constant term

$$f(i\infty)=\frac12\biggl(\frac{1+\zeta_N^b}{1-\zeta_N^b}\,\frac{1+\zeta_N^{bc}}{1-\zeta_N^{bc}}-\frac{1+\zeta_N^a}{1-\zeta_N^a}\,\frac{1+\zeta_N^{ac}}{1-\zeta_N^{ac}}\biggr)$$
$$=-\frac12\biggl(\cot\frac{\pi b}{N}\,\cot\frac{\pi bc}{N}-\cot\frac{\pi a}{N}\,\cot\frac{\pi ac}{N}\biggr)$$

in the q-expansion $f(\tau)=f(i\infty)+\sum_{n\ge1}c_nq^n$. Furthermore, if a linear combination

$$f(\tau)=\sum_{(a,b,c)\in\mathcal M}\lambda_{a,b,c}f_{a,b;c}(\tau),\qquad \lambda_{a,b,c}\in\mathbb C,$$

happens to be a *cusp* form (and this corresponds to application of Theorem 1 to Mahler measures), then formula (6) produces the evaluation

$$\sum_{(a,b,c)\in\mathcal M}\lambda_{a,b,c}\int_{c/N}^{i\infty}\eta(g_a,g_b)=\frac{1}{4\pi}\,L(f(\tau),2).$$

Note as well that the theorem allows one to integrate between any cusps c/N and d/N with the help of $\int_{c/N}^{d/N}=\int_{c/N}^{i\infty}-\int_{d/N}^{i\infty}$.

Here is a sketch of the proof of Theorem 1; details are given in Section 2. We parameterise the contour of integration by $\tau=c/N+it$, $0<t<\infty$, and note that the Möbius transformation $\tau':=(c\tau-(c^2+1)/N)/(N\tau-c)$ preserves the contour: $\tau'=c/N+i/(N^2t)$. Then the logarithms of $g_a(\tau)$ and $g_b(\tau)$, hence their real and imaginary parts — everything we need for computing the form (1), can be written as explicit Eisenstein series of weight 0 in powers of $\exp(-2\pi t)$ and $\exp(-2\pi/(N^2t))$. Finally, executing an analytical change of variable from [14] (as detailed in [18, Section 3]) the integrand becomes a linear combination of pairwise products of weight 1 Eisenstein series in powers of $\exp(-2\pi t)$ integrated against the form $t\,\mathrm dt$ along the line $0<t<\infty$.

Applications of Theorem 1 to Boyd's and Rodriguez-Villegas' conjectural evaluations of 2-variable Mahler measures are discussed in Section 3, while Section 4 highlights some open problems related to 3-variable Mahler measures.

2. Proof of the Mellit–Brunault formula

The two auxiliary lemmas indicate particular modular transformations of the modular functions (5) and the Eisenstein series (7). Lemma 1 also describes the asymptotic behaviour of the modular functions (5) in a neighbourhood of a cusp with $\operatorname{Re}\tau=0$; it is used in the form (10) to determine the integration contours (4) for our applications in Section 3.

Lemma 1. *For a,c integers,*

$$
\begin{aligned}
\log g_a(c/N+it) &= \pi icB(a/N)-\pi t\,NB(a/N) \\
&\qquad -\sum_{\substack{m,n\ge1\\ n\equiv a}}\frac{\zeta_N^{acm}}{m}\exp(-2\pi mnt)-\sum_{\substack{m,n\ge1\\ n\equiv -a}}\frac{\zeta_N^{-acm}}{m}\exp(-2\pi mnt) \\
&= -\frac{\pi i}{2}+\pi ia(c^2+1)(N-ac)+\pi icB(ac/N)-\frac{\pi B(ac/N)}{Nt} \\
&\qquad -\sum_{\substack{m,n\ge1\\ n\equiv ac}}\frac{\zeta_N^{-am}}{m}\exp\biggl(-\frac{2\pi mn}{N^2t}\biggr)-\sum_{\substack{m,n\ge1\\ n\equiv -ac}}\frac{\zeta_N^{am}}{m}\exp\biggl(-\frac{2\pi mn}{N^2t}\biggr),
\end{aligned}
$$

where $t>0$.

Proof. First note that definition (5) implies

$$
\begin{aligned}
\log g_a(\tau) &= \pi i\tau\,NB(a/N)+\sum_{\substack{n\ge1\\ n\equiv a}}\log(1-q^n)+\sum_{\substack{n\ge1\\ n\equiv -a}}\log(1-q^n) \\
&= \pi i\tau\,NB(a/N)-\sum_{\substack{m,n\ge1\\ n\equiv a}}\frac{q^{mn}}{m}-\sum_{\substack{m,n\ge1\\ n\equiv -a}}\frac{q^{mn}}{m}.
\end{aligned}
$$

Therefore, the substitution $\tau=c/N+it$, equivalently $q=\zeta_N^c\exp(-2\pi t)$, results in the first expansion of the lemma.

Secondly, the modular units (5) are particular cases of the 'generalized Dedekind eta functions' [17, eq. (3)]. Applying [17, Theorem 1] with the choice $h=0$ and $\gamma=\left(\begin{smallmatrix} c & -c^2-1\\ 1 & -c\end{smallmatrix}\right)$ we deduce that

$$
g_a(\tau)=\widetilde g_{a,c}\biggl(\frac{c\tau-(c^2+1)/N}{N\tau-c}\biggr),
$$

where

$$
\begin{aligned}
\widetilde g_{a,c}(\tau) &:= \exp(-\pi i/2+\pi ia(c^2+1)(N-ac))\,q^{NB(ac/N)/2} \\
&\qquad\times\prod_{\substack{n\ge1\\ n\equiv ac \bmod N}}(1-\zeta_N^{-a(c^2+1)}q^n)\prod_{\substack{n\ge1\\ n\equiv -ac \bmod N}}(1-\zeta_N^{a(c^2+1)}q^n).
\end{aligned}
$$

On the other hand,

$$
\tau' := \frac{c\tau-(c^2+1)/N}{N\tau-c}\bigg|_{\tau=c/N+it}=\frac{c}{N}+\frac{i}{N^2t},
$$

so that

$$
\begin{aligned}
\log\widetilde g_{a,c}(\tau') &= -\frac{\pi i}{2}+\pi ia(c^2+1)(N-ac)+\pi icB(ac/N)-\frac{\pi B(ac/N)}{Nt} \\
&\quad -\sum_{\substack{m,n\ge1\\ n\equiv ac}}\frac{\zeta_N^{-a(c^2+1)m+cmn}}{m}\exp\biggl(-\frac{2\pi mn}{N^2t}\biggr)-\sum_{\substack{m,n\ge1\\ n\equiv -ac}}\frac{\zeta_N^{a(c^2+1)m+cmn}}{m}\exp\biggl(-\frac{2\pi mn}{N^2t}\biggr),
\end{aligned}
$$

and it remains to use the congruences $n\equiv ac$ and $n\equiv -ac$ to simplify the exponents of the roots of unity. □

Lemma 2. *For a,b integers not divisible by N,*

$$\frac{1}{N^2\tau}\,e_{a,b}\Big(-\frac{1}{N^2\tau}\Big)=\widetilde{e}_{a,b}(\tau):=\sum_{\substack{m,n\ge1\\ m\equiv a,\ n\equiv b}}q^{mn}-\sum_{\substack{m,n\ge1\\ m\equiv -a,\ n\equiv -b}}q^{mn}.$$

Proof. In [16, Section 7] the following general Eisenstein series of weight 1 and level N are introduced:

$$G_{a,c}(\tau)=G_{N,1;(c,a)}(\tau):=-\frac{2\pi i}{N}\Bigg(\kappa_{a,c}+\sum_{\substack{m,n\ge1\\ n\equiv c \bmod N}}\zeta_N^{am}q^{mn/N}-\sum_{\substack{m,n\ge1\\ n\equiv -c \bmod N}}\zeta_N^{-am}q^{mn/N}\Bigg),$$

where

$$\kappa_{a,c}:=\begin{cases}\dfrac12\,\dfrac{1+\zeta_N^a}{1-\zeta_N^a} & \text{if } c\equiv 0 \bmod N,\\[2mm] \dfrac12-\Big\{\dfrac{c}{N}\Big\} & \text{if } c\not\equiv 0 \bmod N.\end{cases}$$

Then for $\gamma=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in SL_2(\mathbb{Z})$ we have

$$G_{a,c}(\gamma\tau)=(C\tau+D)G_{aD+cB,aC+cA}(\tau). \tag{8}$$

The partial Fourier transform from [8, Chapter III] applied to $G_{a,c}$ results in

$$\begin{aligned}\widehat{G}_{a,b}(\tau):=\sum_{c=0}^{N-1}\zeta_N^{bc}\,G_{a,c}(\tau)&=-\frac{\pi i}{N}\Big(\frac{1+\zeta_N^a}{1-\zeta_N^a}+\frac{1+\zeta_N^b}{1-\zeta_N^b}\Big)\\ &\quad-\frac{2\pi i}{N}\sum_{m,n\ge1}(\zeta_N^{am+bn}-\zeta_N^{-(am+bn)})q^{mn/N}.\end{aligned}$$

On the other hand, taking $\gamma=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ in (8) we find that

$$\begin{aligned}\tau^{-1}\widehat{G}_{a,b}(-1/\tau)&=\sum_{c=0}^{N-1}\zeta_N^{bc}\,G_{-c,a}(\tau)\\ &=-\frac{2\pi i}{N}\sum_{c=0}^{N-1}\zeta_N^{bc}\Bigg(\frac12-\Big\{\frac{a}{N}\Big\}+\sum_{\substack{m,n\ge1\\ n\equiv a}}\zeta_N^{-cm}q^{mn/N}-\sum_{\substack{m,n\ge1\\ n\equiv -a}}\zeta_N^{cm}q^{mn/N}\Bigg)\\ &=-2\pi i\Bigg(\sum_{\substack{m,n\ge1\\ n\equiv a,\ m\equiv b}}q^{mn/N}-\sum_{\substack{m,n\ge1\\ n\equiv -a,\ m\equiv -b}}q^{mn/N}\Bigg).\end{aligned}$$

Using now $\widehat{G}_{a,b}(N\tau)=-2\pi i\,e_{a,b}(\tau)/N$ we obtain the desired transformation. □

The next two statements are to take care of integrating the constant terms of auxiliary Eisenstein series.

Lemma 3. *For a, b integers not divisible by N,*

$$\int_0^\infty \left(e_{a,b}(it) + e_{a,-b}(it) - \frac{1+\zeta_N^a}{1-\zeta_N^a}\right) t\,\mathrm{d}t = i\,\mathrm{Cl}_2\left(\frac{2\pi a}{N}\right) B\left(\frac{b}{N}\right),$$

where

$$\mathrm{Cl}_2(x) := \sum_{m\ge1} \frac{\sin mx}{m^2}$$

denotes Clausen's (dilogarithmic) function.

Proof. The integral under consideration is equal to

$$\begin{aligned}&\int_0^\infty \sum_{m,n\ge1} (\zeta_N^{am+bn} - \zeta_N^{-(am+bn)} + \zeta_N^{am-bn} - \zeta_N^{-(am-bn)}) \exp(-2\pi mnt)\,t\,\mathrm{d}t \\ &\quad = \int_0^\infty \sum_{m,n\ge1} (\zeta_N^{am} - \zeta_N^{-am})(\zeta_N^{bn} + \zeta_N^{-bn}) \exp(-2\pi mnt)\,t\,\mathrm{d}t.\end{aligned}$$

On using the Mellin transform

$$\int_0^\infty \exp(-2\pi kt) t^{s-1}\,\mathrm{d}t = \frac{\Gamma(s)}{(2\pi)^s k^s} \qquad \text{for} \quad \operatorname{Re} s > 0, \tag{9}$$

the integral of the double sum evaluates to

$$\frac{1}{4\pi^2} \sum_{m\ge1} \frac{\zeta_N^{am} - \zeta_N^{-am}}{m^2} \sum_{n\ge1} \frac{\zeta_N^{bn} + \zeta_N^{-bn}}{n^2} = \frac{i}{\pi^2}\,\mathrm{Cl}_2\left(\frac{2\pi a}{N}\right) \sum_{n\ge1} \frac{\cos(2\pi nb/N)}{n^2}.$$

It remains to use

$$\sum_{n\ge1} \frac{\cos nx}{n^2} = \pi^2 B\left(\frac{x}{2\pi}\right),$$

and the required evaluation follows. ∎

Lemma 4. *For a, b integers not divisible by N,*

$$\begin{aligned}&\int_0^\infty \frac{1}{iNt}\,\mathrm{d} \sum_{m\ge1} \frac{\zeta_N^{am} - \zeta_N^{-am}}{m} \left(\sum_{\substack{n\ge1\\ n\equiv b}} - \sum_{\substack{n\ge1\\ n\equiv -b}}\right) \exp\left(-\frac{2\pi mn}{N^2 t}\right) \\ &\quad = -i\,\mathrm{Cl}_2\left(\frac{2\pi a}{N}\right) \frac{1+\zeta_N^b}{1-\zeta_N^b}.\end{aligned}$$

Proof. Performing the change of variable $u = 1/(N^2 t)$ in the integral, it becomes equal to

$$\frac{2\pi N}{i} \int_0^\infty \sum_{m\ge1} (\zeta_N^{am} - \zeta_N^{-am}) \left(\sum_{\substack{n\ge1\\ n\equiv b}} - \sum_{\substack{n\ge1\\ n\equiv -b}}\right) n \exp(-2\pi mnu)\,u\,\mathrm{d}u,$$

and applying (9) with $s\to 2^+$ it evaluates to

$$\frac{N}{\pi}\sum_{m\ge1}\frac{\sin(2\pi am/N)}{m^2}\lim_{s\to1^+}\biggl(\sum_{\substack{n\ge1\\ n\equiv b}}-\sum_{\substack{n\ge1\\ n\equiv -b}}\biggr)\frac{1}{n^s}$$
$$=\frac{1}{\pi}\operatorname{Cl}_2\biggl(\frac{2\pi a}{N}\biggr)\cdot\bigl(\psi(1-\{b/N\})-\psi(\{b/N\})\bigr)=\frac{1}{\pi}\operatorname{Cl}_2\biggl(\frac{2\pi a}{N}\biggr)\pi\cot\frac{\pi b}{N},$$

where $\psi(x)$ is the logarithmic derivative of the gamma function. It remains to use $\cot(\pi b/N)=-i(1+\zeta_N^b)/(1-\zeta_N^b)$. □

Proof of Theorem 1. To integrate the 1-form $\eta(g_a,g_b)$ along the interval $\tau\in(c/N,i\infty)$ we make the substitution $\tau=c/N+it$, $0<t<\infty$. It follows from Lemma 1 that

$$\log|g_a(\tau)|=-\frac{\pi B(ac/N)}{Nt}-\frac{1}{2}\sum_{m\ge1}\frac{\zeta_N^{am}+\zeta_N^{-am}}{m}\biggl(\sum_{\substack{n\ge1\\ n\equiv ac}}+\sum_{\substack{n\ge1\\ n\equiv -ac}}\biggr)\exp\biggl(-\frac{2\pi mn}{N^2t}\biggr) \tag{10}$$

and

$$\mathrm{d}\arg g_a(\tau)=-\frac{1}{2i}\,\mathrm{d}\sum_{m\ge1}\frac{\zeta_N^{acm}-\zeta_N^{-acm}}{m}\biggl(\sum_{\substack{n\ge1\\ n\equiv a}}-\sum_{\substack{n\ge1\\ n\equiv -a}}\biggr)\exp(-2\pi mnt)$$
$$=\frac{1}{2i}\,\mathrm{d}\sum_{m\ge1}\frac{\zeta_N^{am}-\zeta_N^{-am}}{m}\biggl(\sum_{\substack{n\ge1\\ n\equiv ac}}-\sum_{\substack{n\ge1\\ n\equiv -ac}}\biggr)\exp\biggl(-\frac{2\pi mn}{N^2t}\biggr).$$

This computation implies

$$\eta(g_a,g_b)=-\frac{\pi B(ac/N)}{2iNt}\,\mathrm{d}\sum_{m\ge1}\frac{\zeta_N^{bm}-\zeta_N^{-bm}}{m}\biggl(\sum_{\substack{n\ge1\\ n\equiv bc}}-\sum_{\substack{n\ge1\\ n\equiv -bc}}\biggr)\exp\biggl(-\frac{2\pi mn}{N^2t}\biggr)$$
$$+\frac{1}{4i}\sum_{m_1\ge1}\frac{\zeta_N^{am_1}+\zeta_N^{-am_1}}{m_1}\biggl(\sum_{\substack{n_1\ge1\\ n_1\equiv ac}}+\sum_{\substack{n_1\ge1\\ n_1\equiv -ac}}\biggr)\exp\biggl(-\frac{2\pi m_1n_1}{N^2t}\biggr)$$
$$\times\,\mathrm{d}\sum_{m_2\ge1}\frac{\zeta_N^{bcm_2}-\zeta_N^{-bcm_2}}{m_2}\biggl(\sum_{\substack{n_2\ge1\\ n_2\equiv b}}-\sum_{\substack{n_2\ge1\\ n_2\equiv -b}}\biggr)\exp(-2\pi m_2n_2t)$$

$$
+\frac{\pi B(bc/N)}{2iNt}\,\mathrm{d}\sum_{m\ge1}\frac{\zeta_N^{am}-\zeta_N^{-am}}{m}\biggl(\sum_{\substack{n\ge1\\ n\equiv ac}}-\sum_{\substack{n\ge1\\ n\equiv -ac}}\biggr)\exp\Bigl(-\frac{2\pi mn}{N^2t}\Bigr)
$$

$$
-\frac{1}{4i}\sum_{m_1\ge1}\frac{\zeta_N^{bm_1}+\zeta_N^{-bm_1}}{m_1}\biggl(\sum_{\substack{n_1\ge1\\ n_1\equiv bc}}+\sum_{\substack{n_1\ge1\\ n_1\equiv -bc}}\biggr)\exp\Bigl(-\frac{2\pi m_1n_1}{N^2t}\Bigr)
$$

$$
\times\,\mathrm{d}\sum_{m_2\ge1}\frac{\zeta_N^{acm_2}-\zeta_N^{-acm_2}}{m_2}\biggl(\sum_{\substack{n_2\ge1\\ n_2\equiv a}}-\sum_{\substack{n_2\ge1\\ n_2\equiv -a}}\biggr)\exp(-2\pi m_2n_2t).
$$

The terms involving double sums only can be integrated with the help of Lemma 4, and we obtain

$$
\int_{c/N}^{i\infty}\eta(g_a,g_b)=\frac{\pi i}{2}\,\frac{1+\zeta_N^{bc}}{1-\zeta_N^{bc}}\,\mathrm{Cl}_2\Bigl(\frac{2\pi b}{N}\Bigr)B\Bigl(\frac{ac}{N}\Bigr)-\frac{\pi i}{2}\,\frac{1+\zeta_N^{ac}}{1-\zeta_N^{ac}}\,\mathrm{Cl}_2\Bigl(\frac{2\pi a}{N}\Bigr)B\Bigl(\frac{bc}{N}\Bigr)
$$

$$
-\frac{\pi}{2i}\Biggl(\sum_{m_1,m_2\ge1}(\zeta_N^{am_1}+\zeta_N^{-am_1})(\zeta_N^{bcm_2}-\zeta_N^{-bcm_2})\biggl(\sum_{\substack{n_1\ge1\\ n_1\equiv ac}}+\sum_{\substack{n_1\ge1\\ n_1\equiv -ac}}\biggr)\biggl(\sum_{\substack{n_2\ge1\\ n_2\equiv b}}-\sum_{\substack{n_2\ge1\\ n_2\equiv -b}}\biggr)
$$

$$
-\sum_{m_1,m_2\ge1}(\zeta_N^{bm_1}+\zeta_N^{-bm_1})(\zeta_N^{acm_2}-\zeta_N^{-acm_2})\biggl(\sum_{\substack{n_1\ge1\\ n_1\equiv bc}}+\sum_{\substack{n_1\ge1\\ n_1\equiv -bc}}\biggr)\biggl(\sum_{\substack{n_2\ge1\\ n_2\equiv a}}-\sum_{\substack{n_2\ge1\\ n_2\equiv -a}}\biggr)\Biggr)
$$

$$
\times\frac{n_2}{m_1}\int_0^\infty\exp\Bigl(-2\pi\Bigl(\frac{m_1n_1}{N^2t}+m_2n_2t\Bigr)\Bigr)\,\mathrm{d}t.
$$

Now we execute the change of variable $u=n_2t/m_1$, interchange integration and quadruple summation and use Lemma 2:

$$
\int_{c/N}^{i\infty}\eta(g_a,g_b)=\frac{\pi i}{2}\,\frac{1+\zeta_N^{bc}}{1-\zeta_N^{bc}}\,\mathrm{Cl}_2\Bigl(\frac{2\pi b}{N}\Bigr)B\Bigl(\frac{ac}{N}\Bigr)-\frac{\pi i}{2}\,\frac{1+\zeta_N^{ac}}{1-\zeta_N^{ac}}\,\mathrm{Cl}_2\Bigl(\frac{2\pi a}{N}\Bigr)B\Bigl(\frac{bc}{N}\Bigr)
$$

$$
-\frac{\pi}{2i}\int_0^\infty\sum_{m_1,m_2\ge1}(\zeta_N^{am_1}+\zeta_N^{-am_1})(\zeta_N^{bcm_2}-\zeta_N^{-bcm_2})\exp(-2\pi m_1m_2u)
$$

$$
\times\biggl(\sum_{\substack{n_1\ge1\\ n_1\equiv ac}}+\sum_{\substack{n_1\ge1\\ n_1\equiv -ac}}\biggr)\biggl(\sum_{\substack{n_2\ge1\\ n_2\equiv b}}-\sum_{\substack{n_2\ge1\\ n_2\equiv -b}}\biggr)\exp\Bigl(-\frac{2\pi n_1n_2}{N^2u}\Bigr)
$$

$$
-\sum_{m_1,m_2\ge1}(\zeta_N^{bm_1}+\zeta_N^{-bm_1})(\zeta_N^{acm_2}-\zeta_N^{-acm_2})\exp(-2\pi m_1m_2u)
$$

$$
\times\biggl(\sum_{\substack{n_1\ge1\\ n_1\equiv bc}}+\sum_{\substack{n_1\ge1\\ n_1\equiv -bc}}\biggr)\biggl(\sum_{\substack{n_2\ge1\\ n_2\equiv a}}-\sum_{\substack{n_2\ge1\\ n_2\equiv -a}}\biggr)\exp\Bigl(-\frac{2\pi n_1n_2}{N^2u}\Bigr)\,\mathrm{d}u
$$

$$
\begin{aligned}
&= \frac{\pi i}{2}\,\frac{1+\zeta_N^{bc}}{1-\zeta_N^{bc}}\,\mathrm{Cl}_2\Big(\frac{2\pi b}{N}\Big)B\Big(\frac{ac}{N}\Big) - \frac{\pi i}{2}\,\frac{1+\zeta_N^{ac}}{1-\zeta_N^{ac}}\,\mathrm{Cl}_2\Big(\frac{2\pi a}{N}\Big)B\Big(\frac{bc}{N}\Big) \\
&\quad - \frac{\pi}{2i}\int_0^\infty \Big(e_{a,bc}(iu) - e_{a,-bc}(iu) - \frac{1+\zeta_N^{bc}}{1-\zeta_N^{bc}}\Big)\big(\widetilde{e}_{b,ac}(i/(N^2u)) + \widetilde{e}_{b,-ac}(i/(N^2u))\big) \\
&\qquad - \Big(e_{b,ac}(iu) - e_{b,-ac}(iu) - \frac{1+\zeta_N^{ac}}{1-\zeta_N^{ac}}\Big)\big(\widetilde{e}_{a,bc}(i/(N^2u)) + \widetilde{e}_{a,-bc}(i/(N^2u))\big)\,\mathrm{d}u \\
&= \frac{\pi i}{2}\,\frac{1+\zeta_N^{bc}}{1-\zeta_N^{bc}}\,\mathrm{Cl}_2\Big(\frac{2\pi b}{N}\Big)B\Big(\frac{ac}{N}\Big) - \frac{\pi i}{2}\,\frac{1+\zeta_N^{ac}}{1-\zeta_N^{ac}}\,\mathrm{Cl}_2\Big(\frac{2\pi a}{N}\Big)B\Big(\frac{bc}{N}\Big) \\
&\quad + \frac{\pi}{2}\int_0^\infty \Big(e_{a,bc}(iu) - e_{a,-bc}(iu) - \frac{1+\zeta_N^{bc}}{1-\zeta_N^{bc}}\Big)\big(e_{b,ac}(iu) + e_{b,-ac}(iu)\big)\,u \\
&\qquad - \Big(e_{b,ac}(iu) - e_{b,-ac}(iu) - \frac{1+\zeta_N^{ac}}{1-\zeta_N^{ac}}\Big)\big(e_{a,bc}(iu) + e_{a,-bc}(iu)\big)\,u\,\mathrm{d}u \\
&= \frac{\pi i}{2}\,\frac{1+\zeta_N^{bc}}{1-\zeta_N^{bc}}\,\mathrm{Cl}_2\Big(\frac{2\pi b}{N}\Big)B\Big(\frac{ac}{N}\Big) - \frac{\pi i}{2}\,\frac{1+\zeta_N^{ac}}{1-\zeta_N^{ac}}\,\mathrm{Cl}_2\Big(\frac{2\pi a}{N}\Big)B\Big(\frac{bc}{N}\Big) \\
&\quad + \pi\int_0^\infty \big(e_{a,bc}(iu)e_{b,-ac}(iu) - e_{a,-bc}(iu)e_{b,ac}(iu)\big)\,u \\
&\qquad - \frac{1}{2}\Big(\frac{1+\zeta_N^{bc}}{1-\zeta_N^{bc}}\big(e_{b,ac}(iu) + e_{b,-ac}(iu)\big) - \frac{1+\zeta_N^{ac}}{1-\zeta_N^{ac}}\big(e_{a,bc}(iu) + e_{a,-bc}(iu)\big)\Big)u\,\mathrm{d}u
\end{aligned}
$$

(we apply Lemma 3)

$$
= \pi\int_0^\infty \Big(f_{a,b;c}(iu) + \frac{1}{2}\,\frac{1+\zeta_N^{a}}{1-\zeta_N^{a}}\,\frac{1+\zeta_N^{ac}}{1-\zeta_N^{ac}} - \frac{1}{2}\,\frac{1+\zeta_N^{b}}{1-\zeta_N^{b}}\,\frac{1+\zeta_N^{bc}}{1-\zeta_N^{bc}}\Big)\,u\,\mathrm{d}u,
$$

and the result follows by appealing to (9). □

3. Applications

The modularity theorem guarantees that an *elliptic* curve $C : P(x,y) = 0$ can be parameterised by modular functions $x(\tau)$ and $y(\tau)$, whose level N is necessarily the conductor of C, such that the pull-back of the canonical differential on C is proportional to $2\pi i f(\tau)\,\mathrm{d}\tau = f(\tau)\,\mathrm{d}q/q$, where f is (up to an isogeny) a normalised newform of weight 2 and level N, which automatically happens to be a cusp form and a Hecke eigenform. Computing the conductor of C and producing the cusp form f of this level give an efficient strategy to determine successively the coefficients in the q-expansions of $x(\tau) = \varepsilon_1 q^{-M_1} + \cdots$ and $y(\tau) = \varepsilon_2 q^{-M_2} + \cdots$ subject to $P(x(\tau), y(\tau)) = 0$, where ε_1 and ε_2 are suitable nonzero constants. The particular form of q-expansions only fixes a normalisation of $x(\tau)$ and $y(\tau)$ up to the action of the corresponding congruence subgroup $\Gamma_0(N)$. Finally, it remains to verify whether $x(\tau)$ and $y(\tau)$ just found are modular units — modular functions whose all zeroes and poles are at cusps (so that they admit eta-like product expansions); if this is the case, we can use Theorem 1 to compute the Mahler measure $\mathrm{m}(P(x,y))$. Note that the property of being a modular unit imposes a strong condition on the q-expansion of the logarithmic derivative — it can be easily detected in practice by examining a couple of (hundred) terms in the q-expansion of the latter.

In this section we touch the 'classical' family of Mahler measures

$$\mathrm{m}(xy^2+(x^2+kx+1)y+x)=\mathrm{m}\Big(k+x+\frac1x+y+\frac1y\Big), \qquad k^2\in\mathbb{Z}\setminus\{0,16\},$$

which goes back to the works [3, 7, 13]. Namely, we will see that Theorem 1 applies in the cases when the corresponding zero locus

$$E: k+x+\frac1x+y+\frac1y=0 \tag{11}$$

can be parameterised by modular units. For this family of tempered Laurent polynomials, equation (3) assumes the form

$$\mathrm{m}\Big(k+x+\frac1x+y+\frac1y\Big)=\mathrm{m}(y^2+(k+x+x^{-1})y+1)=\frac{1}{2\pi}r(\{x,y\})([\gamma]), \tag{12}$$

where γ is a single closed path on $E\setminus\{(0,0)\}$ corresponding to the zero $y_1(x)$ of $y^2+(k+x+x^{-1})y+1$ which satisfies $|y_1(x)|\ge 1$.

The above general strategy restricted to the family (11) was identified by Mellit in [11] and illustrated by him on the example of $k=2i$; this is Example 2 below. The modular functions x and y satisfying (11) are searched in the form $x(\tau)=(\varepsilon q)^{-1}+\cdots$ and $y(\tau)=-(\varepsilon q)^{-1}+\cdots$, where $\varepsilon\in\mathbb{Z}[k]$ is chosen so that k/ε is a positive integer. The condition on the pull-back of the canonical differential on E takes the form

$$\frac{q\,(\mathrm{d}x/\mathrm{d}q)}{\varepsilon x(y-1/y)}=f,$$

where $f(\tau)$ is the corresponding Hecke eigenform of weight 2.

The computational part of the examples below was accomplished in `sage` and `gp-pari`, which allowed us to compute as many terms in the q-expansions of a modular parameterisation of a given elliptic curve as requested. Assisted with this software, we were normally able to relate occurring *modular* forms and functions (for example, their product expansions) by computing and examining sufficiently many terms in their q-expansions.

Below we will have occasional appearance of Dedekind's eta-function $\eta(\tau):=q^{1/24}\prod_{n=1}^\infty(1-q^n)$. We hope that this extra eta notation does not cause any confusion with (1), as it depends here on a *single* variable, which is always a rational multiple of τ from the upper halfplane.

Example 1. The most classical example corresponds to the choice $k=1$, when the elliptic curve in (11) has conductor $N=15$ and can be parameterised by modular units

$$\begin{aligned} x(\tau)&=\frac1q\prod_{n=0}^\infty\frac{(1-q^{15n+7})(1-q^{15n+8})}{(1-q^{15n+2})(1-q^{15n+13})}=\frac{g_7(\tau)}{g_2(\tau)},\\ y(\tau)&=-\frac1q\prod_{n=0}^\infty\frac{(1-q^{15n+4})(1-q^{15n+11})}{(1-q^{15n+1})(1-q^{15n+14})}=-\frac{g_4(\tau)}{g_1(\tau)},\end{aligned}$$

so that

$$\frac{q\,(\mathrm{d}x/\mathrm{d}q)}{x(y-1/y)} = f_{15}(\tau) := \eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau)$$

and the path of integration γ in (12) corresponds to the range of τ between the two cusps $-1/5$ and $1/5$ of $\Gamma_0(15)$. Therefore, Theorem 1 results in

$$\begin{aligned}\mathrm{m}\Big(1+x+\frac{1}{x}+y+\frac{1}{y}\Big) &= \frac{1}{2\pi}\Big(\int_{-1/5}^{i\infty} - \int_{1/5}^{i\infty}\Big)\eta(g_7/g_2, g_4/g_1)\\ &= \frac{1}{8\pi^2}L(2f_{7,4;-3} - 2f_{7,1;-3} - 2f_{2,4;-3} + 2f_{2,1;-3}, 2)\\ &= \frac{15}{4\pi^2}L(f_{15}, 2),\end{aligned}$$

which is precisely Boyd's conjecture from [3] first proven in [15].

Note that this evaluation implies some other Mahler measures, namely [9, 10]

$$\begin{aligned}\mathrm{m}\Big(5+x+\frac{1}{x}+y+\frac{1}{y}\Big) &= 6\mathrm{m}\Big(1+x+\frac{1}{x}+y+\frac{1}{y}\Big)\\ \mathrm{m}\Big(16+x+\frac{1}{x}+y+\frac{1}{y}\Big) &= 11\mathrm{m}\Big(1+x+\frac{1}{x}+y+\frac{1}{y}\Big),\\ \mathrm{m}\Big(3i+x+\frac{1}{x}+y+\frac{1}{y}\Big) &= 5\mathrm{m}\Big(1+x+\frac{1}{x}+y+\frac{1}{y}\Big),\end{aligned}$$

though the corresponding elliptic curves $k+x+1/x+y+1/y=0$ for $k=5$, 16 and $3i$ are not parameterised by modular units.

Example 2 ([11])**.** The modular parameterisation of (11) for $k=2i$ (the conductor of elliptic curve is then $N=40$) and the corresponding Mahler measure evaluation

$$\mathrm{m}\Big(2i+x+\frac{1}{x}+y+\frac{1}{y}\Big) = \frac{10}{\pi^2}L(f_{40}, 2),$$

where

$$f_{40}(\tau) := \frac{\eta(\tau)\eta(8\tau)\eta(10\tau)^2\eta(20\tau)^2}{\eta(5\tau)\eta(40\tau)} + \frac{\eta(2\tau)^2\eta(4\tau)^2\eta(5\tau)\eta(40\tau)}{\eta(\tau)\eta(8\tau)},$$

were given in Mellit's talk [11]. He identifies $x(\tau)$ and $y(\tau)$ with infinite products which are fully expressible by means of Ramanujan's lambda function

$$\lambda(\tau) = q^{1/5}\prod_{n=1}^{\infty}(1-q^n)^{\left(\frac{n}{5}\right)} = q^{1/5}\prod_{n=1}^{\infty}\frac{(1-q^{5n-1})(1-q^{5n-4})}{(1-q^{5n-2})(1-q^{5n-3})};$$

namely,

$$\begin{aligned}x(\tau) &= -i\,\frac{\lambda(4\tau)}{\lambda(\tau)\lambda(8\tau)} = -i\,\frac{g_2g_3g_7g_{13}g_{16}g_{17}g_{18}}{g_1g_6g_8g_9g_{11}g_{14}g_{19}},\\ y(\tau) &= i\,\frac{\lambda(\tau)\lambda(2\tau)}{\lambda(8\tau)} = i\,\frac{g_1g_9g_{11}g_{16}g_{19}}{g_3g_7g_8g_{13}g_{17}}\end{aligned}$$

in the notation (5) with $N=40$. The corresponding range of τ for the path γ in (12) is from $1/10$ to $-2/5$.

Example 3. The elliptic curve (11) for $k = 2$ has conductor $N = 24$ and admits parameterisation by modular units

$$x(\tau) = \frac{g_1 g_{10} g_{11}}{g_2 g_5 g_7}, \qquad y(\tau) = -\frac{g_5 g_7}{g_1 g_{11}}.$$

Theorem 1 applies and produces the evaluation

$$\begin{aligned}\mathrm{m}\Big(2 + x + \frac{1}{x} + y + \frac{1}{y}\Big) &= \frac{1}{2\pi}\Big(\int_{-1/8}^{i\infty} - \int_{1/8}^{i\infty}\Big)\eta\Big(\frac{g_1 g_{10} g_{11}}{g_2 g_5 g_7}, \frac{g_5 g_7}{g_1 g_{11}}\Big)\\ &= \frac{6}{\pi^2} L(f_{24}, 2),\end{aligned}$$

where $f_{24}(\tau) := \eta(2\tau)\eta(4\tau)\eta(6\tau)\eta(12\tau)$, conjectured in [3] and established in [14].

Example 4. For $N = 17$, the pair of modular units

$$x(\tau) = -i\,\frac{g_2 g_8}{g_1 g_4}, \qquad y(\tau) = i\,\frac{g_6 g_7}{g_3 g_5}$$

parameterise the elliptic curve $i + x + 1/x + y + 1/y = 0$. Applying Theorem 1 for τ ranging from $3/17$ to $-3/17$, we obtain

$$\mathrm{m}\Big(i + x + \frac{1}{x} + y + \frac{1}{y}\Big) = \frac{17}{2\pi^2} L(f_{17}, 2),$$

where

$$\begin{aligned}f_{17}(\tau) := \frac{q\,(\mathrm{d}x/\mathrm{d}q)}{ix(y - 1/y)} &= q - q^2 - q^4 - 2q^5 + 4q^7 + 3q^8 - 3q^9 + 2q^{10}\\ &\quad - 2q^{13} - 4q^{14} - q^{16} + q^{17} + O(q^{18}).\end{aligned}$$

This Mahler measure evaluation was conjectured in [13, Table 4].

Example 5. Another conjecture in [13, Table 4],

$$\mathrm{m}\Big(\sqrt{2} + x + \frac{1}{x} + y + \frac{1}{y}\Big) = \frac{7}{2\pi^2} L(f_{56}, 2),$$

corresponds to $k = \sqrt{2}$ in (11) and an elliptic curve over $\mathbb{Z}$ of conductor $N = 56$. The conjecture follows from parameterisation of the curve by the couple

$$\begin{aligned}x(\tau) &= \frac{1}{\sqrt{2}}\,\frac{\eta(\tau)\eta(4\tau)^2\eta(7\tau)\eta(28\tau)^2}{\eta(2\tau)^2\eta(8\tau)\eta(14\tau)^2\eta(56\tau)},\\ y(\tau) &= -\frac{1}{\sqrt{2}}\,\frac{\eta(2\tau)\eta(4\tau)\eta(14\tau)\eta(28\tau)}{\eta(\tau)\eta(7\tau)\eta(8\tau)\eta(56\tau)},\end{aligned}$$

so that

$$\begin{aligned}f_{56}(\tau) := \frac{q\,(\mathrm{d}x/\mathrm{d}q)}{\sqrt{2}\,x(y - 1/y)} &= q + 2q^5 - q^7 - 3q^9 - 4q^{11} + 2q^{13} - 6q^{17} + 8q^{19}\\ &\quad - q^{25} + 6q^{29} + 8q^{31} + O(q^{34}),\end{aligned}$$

and integration in Theorem 1 for $\tau \in (-15/56, -7/56) \cup (5/56, 13/56)$.

It is not clear whether there are finitely or infinitely many cases of the parameter k in (11) subject to parameterisation by modular units. A possible approach in cases when such parameterisation is not available is writing down algebraic relations between any two standard modular units (5) of a given level N and sieving the relations which may be used in producing the Mahler measures of 2-variable polynomials which are potentially linked to the wanted Mahler measures by K-theoretic machinery [6, 9, 10].

Finding what curves $C : P(x,y) = 0$ can be parameterised by modular units is an interesting question itself. F. Brunault notices some heuristics to the fact that there are only finitely many function fields F of a given genus g over $\mathbb{Q}$ which embed into the function field of a modular curve such that F can be generated by modular units; for $g \ge 2$ this follows from [1, Conjecture 1.1].

4. 3-variable Mahler measures

It would be desirable to have an analogue of Theorem 1 for 3-variable Mahler measures of (Laurent) polynomials $P(x,y,z)$ such that the intersection of the zero loci $P(x,y,z) = 0$ and $P(1/x,1/y,1/z) = 0$ defines an elliptic curve E, and $\mathrm{m}(P)$ is presumably related to the L-series of E evaluated at $s = 3$. No example of this type is established, and one of the simplest evaluations is Boyd's conjecture [4]

$$\mathrm{m}\big((1+x)(1+y)-z\big) \overset{?}{=} 2L'(E_{15},-1) = \frac{225}{4\pi^4}L(E_{15},3).$$

On the surface $(1+x)(1+y) - z = 0$ we have

$$\begin{aligned} x \wedge y \wedge z &= x \wedge y \wedge (1+x)(1+y) = x \wedge y \wedge (1+x) + x \wedge y \wedge (1+y) \\ &= -x \wedge (1+x) \wedge y + y \wedge (1+y) \wedge x \\ &= -(-x) \wedge (1+x) \wedge y + (-y) \wedge (1+y) \wedge x. \end{aligned}$$

Applying the machinery described in [6, Section 5.2] to the 3-variable polynomial $P(x,y,z) = (1+x)(1+y) - z$ we obtain

$$m(P) = \frac{1}{4\pi^2}\int_\gamma \big(\omega(-x,y) - \omega(-y,x)\big),$$

where

$$\omega(g,h) := D(g)\,\mathrm{d}\arg h + \frac{1}{3}\big(\log|g|\,\mathrm{d}\log|1-g| - \log|1-g|\,\mathrm{d}\log|g|\big)\log|h| \qquad (13)$$

and

$$\begin{aligned} \gamma := \{(x,y,z) : |x| = |y| = |z| = 1\} &\cap \{(x,y,z) : (1+x)(1+y) - z = 0\} \\ &\cap \{(x,y,z) : (1+x)(1+y)z - xy = 0\}. \end{aligned}$$

Note that $\{(1+x)(1+y) - z = 0\} \cap \{(1+x)(1+y)z - xy = 0\}$ is the double cover of an elliptic curve of conductor 15. Indeed, eliminating z we can write (one half of) its equation as

$$(1+x_1^2)(1+y_1^2) + x_1y_1 = 0$$

in variables $x_1 = \sqrt{x}$, $y_1 = \sqrt{y}$, or

$$x_2 + 1/x_2 + y_2 + 1/y_2 + 1 = 0$$

in variables $x_2 = x_1 y_1$, $y_2 = x_1/y_1$. Using the parameterisation of the latter equation by the modular units from Example 1 we find out that

$$m(P) = \frac{1}{2\pi^2} \int_{-1/5}^{1/5} \bigl(\omega(X,Y) - \omega(Y,X)\bigr)$$

where

$$X(\tau) := \frac{g_4(\tau)g_7(\tau)}{g_1(\tau)g_2(\tau)} = q^{-2} + O(q^{-1}) \quad \text{and} \quad Y(\tau) := \frac{g_1(\tau)g_7(\tau)}{g_2(\tau)g_4(\tau)} = 1 + O(q).$$

Also note that

$$1 - X(\tau) = -\frac{g_6(\tau)g_7(\tau)}{g_1(\tau)g_3(\tau)} = -q^{-2} + O(q^{-1}) \quad \text{and} \quad 1 - Y(\tau) = \frac{g_1(\tau)g_3(\tau)}{g_2(\tau)g_6(\tau)} = q + O(q^2)$$

are modular units.

The problem with integrating the form (13) is that it is, roughly speaking, integrating the product of *three* modular components: two of them are logarithms of modular functions (hence of weight 0) and one is the logarithmic derivative of a modular function (hence of weight 2). On the other hand, the expected data for applying the method from [14] used in our proof of Theorem 1 in Section 2 would be integrating a product of *two* Eisenstein series of weights -1 and 3 (see [18] for details).

Acknowledgements. This note would be hardly possible without constant patience of A. Mellit in describing details of his work with F. Brunault. I am pleased to thank Mellit for all those lectures he delivered to me in person and by e-mail, as well as for providing me with the sketch [12] of proof of what is stated here as Theorem 1. I am deeply grateful to F. Brunault, M. Rogers and J. Wan for their helpful assistance on certain parts of this work. Finally, I thank the anonymous referee for her valuable comments and healthy criticism that helped to improve the exposition.

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